International Journal of Theoretical Physics, Vol. 42, No. 6, June 2003 (© 2003)

Symplectic Structure in Brane Mechanics[1]

Brandon Carter[2]

Received April 14, 2003

This paper treats the generalization to brane dynamics of the covariant canonical variational procedure leading to the construction of a conserved bilinear symplectic current in the manner originally developed by Witten, Zuckerman, and others in the context of field theory. After a general presentation, including a review of the relationships between the various (Lagrangian, Eulerian, and other) relevant kinds of variation, the procedure is illustrated by application to the particularly simple case of branes of the Dirac–Goto–Nambu type, in which internal fields are absent.

KEY WORDS: symplectic structure; brane mechanics.

1. INTRODUCTION

The purpose of this paper is to consider the application to classical brane mechanics of the general principles of covariant canonical variational analysis, which provides a canonical symplectic structure, whose potential utility as a starting point for the covariant construction of corresponding quantum systems has been emphasized by Witten, Zuckerman, and others (Cartas-Fuentevilla, 1998; Crncovic and Witten, 1987; Nutku, 2000; Rovelli, 2002; Soh, 1994; Witten, 1986; Zuckerman, 1987) in the context of relativistic field theories. The task of extending such analysis from ordinary field to branes (meaning systems with support confined to a lower-dimensional worldsheet) has recently been taken up by Cartas-Fuentevilla (2002a,b). The necessary analysis has been facilitated by the relatively new development (Battye and Carter, 1995, 2000; Carter, 1993) of suitably covariant methods of geometrical analysis, which have already been shown to be far more efficient than the more cumbersome (and error prone) frame-dependent methods used in earlier work for treating other problems, such as the divergences arising from self-interaction (Carter, 1997; Carter *et al.*, in press; Carter and Battye, 1998).

One of the questions that has arisen in this work is that how the conserved antisymmetric bilinear perturbation current that was obtained by a different approach

[1] An earlier version of this paper was presented at *2002 Peyresq workshop*.

[2] LuTh. Observatoire de Paris, 92 Meudon, Paris; e-mail: brandon.carter@obspm.fr.

in my original perturbation analysis (Carter, 1993) of simple Dirac–Nambu–Goto type branes is related to the closed symmetric structure of the canonical treatment. The claim (Carter-Fuentevilla, 2002a) that both approaches lead ultimately to the same result was based on an argument that, in its original version, depended on intermediate steps involving a questionable logical shortcut, but the ensuing conclusion is fully confirmed by the more rigorous and complete treatment provided here.

2. BRANE VARIATIONAL PRINCIPLE

The present work will be concerned with the very broad category of conservative p-brane models whose mechanical evolution is governed by an action integral of the form

$$\mathcal{I} = \int \mathcal{L}\, d^{p+1}\sigma, \tag{1}$$

over a supporting worldsheet with internal coordinates $\sigma^i (i = 0, 1, \ldots, p)$, and induced metric $\eta_{ij} = g_{\mu\nu} x^\mu,{}_i x^\nu_{,j}$, in a background with coordinates x^μ, $(\mu = 0, 1, \ldots, d)$ $(d \geq p)$ and (flat or curved) space–time metric $g_{\mu\nu}$. The relevant Lagrangian scalar density is expressible in the form

$$\mathcal{L} = \|\eta\|^{1/2} L, \tag{2}$$

where L is scalar function of a set of field components q^{A} (including background coordinates) and of their surface deriatives, $q^{\mathrm{A}}_{,i} = \partial_i q^{\mathrm{A}} = \partial q^{\mathrm{A}}/\partial\sigma^i$. The relevant field variables q^{A} can be of internal or external kind, the most obvious example of the latter kind being the background coordinates x^μ themselves.

The generic action variation,

$$\delta\mathcal{L} = \mathcal{L}_{\mathrm{A}} \delta q^{\mathrm{A}} + p^i_{\mathrm{A}} \delta q^{\mathrm{A}}_{,i}, \tag{3}$$

specifies a set of partial derivative components $\mathcal{L}_{\mathrm{A}}$ and an associated set of generalized momentum components p^i_{A}. According to variation principle, the dynamically admissible "on-shell" configurations are those characterized by the vanishing of the Eulerian derivative as given by

$$\frac{\delta\mathcal{L}}{\delta q^{\mathrm{A}}} = \mathcal{L}_{\mathrm{A}} - p^i_{\mathrm{A},i}. \tag{4}$$

In terms of this Eulerian derivative, the generic Lagrangian variation will have the form

$$\delta\mathcal{L} = \frac{\delta\mathcal{L}}{\delta q^{\mathrm{A}}} \delta q^{\mathrm{A}} + \left(p^i_{\mathrm{A}} \delta q^{\mathrm{A}}\right)_{,i}. \tag{5}$$

There will be a corresponding pseudo-Hamiltonian scalar density

$$\mathcal{H} = p^i_{\rm A} q^{\rm A}_{,i} - \mathcal{L}, \tag{6}$$

for which

$$\delta\mathcal{H} = q^{\rm A}_{,i}\delta p^i_{\rm A} - \mathcal{L}_{\rm A}\delta q^{\rm A}. \tag{7}$$

(The covariance of such a pseudo Hamiltonian distingushes it form the ordinary kind of Hamiltonian, which depends on the introduction of some preferred time foliation.)

For an on-shall configuration, i.e., when the dynamical equations

$$\frac{\delta\mathcal{L}}{\delta q^{\rm A}} = 0, \tag{8}$$

are satisfied, the Lagrangian variation will reduce to a pure surface divergence,

$$\delta\mathcal{L} = \left(p^i_{\rm A}\delta q^{\rm A}\right)_{,i}, \tag{9}$$

and the corresponding on-shell pseudo-Hamiltonian variation will take the form

$$\delta\mathcal{H} = q^{\rm A}_{,i}\delta p^i_{\rm A} - p^i_{{\rm A},i}\delta q^{\rm A}. \tag{10}$$

3. CANONICAL SYMPLECTIC STRUCTURE

It is evident from the preceeding work that the generic first-order variation of the Lagrangian will be expressible as

$$\delta\mathcal{L} = \frac{\delta\mathcal{L}}{\delta q^{\rm A}}\delta q^{\rm A} + \vartheta^i_{\ ,i}, \tag{11}$$

in terms of the generalized Liouville 1-form (on the configuration space cotangent bundle) that is defined by

$$\vartheta^i = p^i_{\rm A}\delta q^{\rm A}, \tag{12}$$

Let us now consider the effect of a pair of successive independent variations $\acute{\delta}$, $\grave{\delta}$, which will give a second-order variation of the form

$$\grave{\delta}\acute{\delta}\mathcal{L} = \grave{\delta}\left(\frac{\delta\mathcal{L}}{\delta q^{\rm A}}\right)\acute{\delta}q^{\rm A} + \frac{\delta\mathcal{L}}{\delta q^{\rm A}}\grave{\delta}\acute{\delta}q^{\rm A} + \left(\grave{\delta}p^i_{\rm A}\acute{\delta}q^{\rm A} + p^i_{\rm A}\grave{\delta}\acute{\delta}q^{\rm A}\right)_{,i}. \tag{13}$$

Thus using the commutation relation $\grave{\delta}\acute{\delta} = \acute{\delta}\grave{\delta}$ one gets

$$\grave{\delta}\left(\frac{\delta\mathcal{L}}{\delta q^{\rm A}}\right)\acute{\delta}q^{\rm A} - \acute{\delta}\left(\frac{\delta\mathcal{L}}{\delta q^{\rm A}}\right)\grave{\delta}q^{\rm A} = \hat{\varpi}^i_{\ ,i}, \tag{14}$$

where the symplectic 2-form (on the configuration space cotangent bundle) is defined by

$$\hat{\varpi}^i = \acute{\delta}p^i_{\rm A}\grave{\delta}q^{\rm A} - \grave{\delta}p^i_{\rm A}\acute{\delta}q^{\rm A}. \tag{15}$$

For an on-shell perturbation we thus obtain

$$\frac{\delta\mathcal{L}}{\delta q^{\mathrm{A}}} = 0 \quad \Rightarrow \quad \delta\mathcal{L} = \vartheta^{i}_{\ ,i}, \tag{16}$$

while for a pair of on-shell perturbations we obtain

$$\acute{\delta}\left(\frac{\delta\mathcal{L}}{\delta q^{\mathrm{A}}}\right) = \grave{\delta}\left(\frac{\delta\mathcal{L}}{\delta q^{\mathrm{A}}}\right) = 0 \quad \Rightarrow \quad \hat{\varpi}^{i}_{\ ,i} = 0. \tag{17}$$

The foregoing surface current conservation law is expressible in shorthand notation as the condition

$$\varpi^{i}_{\ ,i} = 0, \tag{18}$$

in which the closed (since manifestly exact) symplectic 2-form (15) is specified using concise wedge product notation as

$$\varpi^{i} = \delta \wedge \vartheta^{i} = \delta p^{i}_{\mathrm{A}} \wedge \delta q^{\mathrm{A}}. \tag{19}$$

It is to be remarked that some authors prefer to use an even more concise notation system in which it is not just the relevant distinguishing (in our case acute and grave accent) indices that are omitted but even the wedge symbol $\wedge$ that indicates the antisymmetrized product relation. However, such an extreme level abbreviation is dangerous in contexts such as that of the present work in which symmetric products are also involved, as is shown by the example (Cartas-Fuentevilla, 2002a) discussed later, in which a formula involving a symmetric product was applied as if it were an antisymmetric product.

4. TRANSLATION INTO STRICTLY TENSORIAL FORM

In accordance with the strategy (Carter, 1993) of avoiding the supplementary gauge dependence involved in the use of auxiliary structures such as local frames and internal surface coordinates by working as far as possible just with quantities that are strictly tensorial with respect to the background space, it will be preferable for many purposes to translate the surface current densities whose components ϑ^{i} and ϖ^{i} depend on the choice of the internal coordinates σ^{i}, into terms of the corresponding vectorial quantities, which will have strictly tensorial background coordinate components given by

$$\Theta^{\nu} = \|\eta\|^{-1/2} x^{\nu}_{\ ,i} \vartheta^{i}, \qquad \Omega^{\nu} = \|\eta\|^{-1/2} x^{\nu,}_{\ ,i} \varpi^{i}. \tag{20}$$

These currents will have strictly scalar surface divergences given in terms of the corresponding scalar densities by

$$\overline{\nabla}_{\nu}\Theta^{\nu} = \|\eta\|^{-1/2} \vartheta^{i}_{\ ,i}, \qquad \overline{\nabla}_{\nu,}\Omega^{\nu} = \|\eta\|^{-1/2} \varpi^{i}_{\ ,i} \tag{21}$$

where $\bar{\nabla}$ is the surface projected covariant differentiation operator defined in terms of the fundamental tensor $\eta^{\mu\nu} = \eta^{ij} x^{\mu}_{,i} x^{\nu}_{,j}$ by $\bar{\nabla}_{\nu} = \eta^{\mu}_{\nu} \nabla_{\mu}$.

By the preceeding analysis, a Liouville current conservation law of the form

$$\bar{\nabla}_{\nu} \Theta^{\nu} = 0 \tag{22}$$

will hold for any symmetry generating pertubation, i.e., for any infinitesimal variation δq^{A} such that $\delta \mathcal{L} = 0$, and a symplectic current conservation law of the form

$$\bar{\nabla}_{\nu} \Omega^{\nu} = 0 \tag{23}$$

will hold for any pair of perturbations that are on-shell, i.e., such that $\delta(\delta \mathcal{L} / \delta q^{\mathrm{A}}) = 0$.

5. COVARIANT VARIATION FORMULAE

For physical evaluation of quantities of the Liouville and symplectic currents Θ^{μ} and Θ^{μ}, it is often more convenient to work with something less coordinate-gauge-dependent than the simple worldsheet-based field component variations δq^{A} used in preceding work.

In particular, if the field component q^{A} is of a kind that is defined over the background—not just confined to brane worldsheet with internal coordinates σ^i—then with respect to a given system of external coordinates (which might, for example, be of Minkowski type if the background is flat) in terms of which $\partial_i q^{\mathrm{A}} = x^{\mu}_{,i} \partial_{\mu} q^{\mathrm{A}}$, the field will have an *Eulerian* (fixed background) variation $\underset{\mathrm{E}}{\delta}\, q^{\mathrm{A}}$ that is well defined independently of any choice of the internal coordinates σ^i, unlike the simple brane worldsheet variation, which will be given in terms of the relevant *displacement vector*, $\xi^{\mu} = \delta x^{\mu}$, by

$$\delta q^{\mathrm{A}} = \underset{\mathrm{E}}{\delta}\, q^{\mathrm{A}} + \xi^{\mu} \partial_{\mu} q^{\mathrm{A}}. \tag{24}$$

When one is dealing with a background field that is not simply a scalar but of a more general tensorial nature, it will commonly be desirable to go on to convert the *Eulerian variation* formula

$$\underset{\mathrm{E}}{\delta} = \delta - \vec{\xi} \cdot \partial \tag{25}$$

into terms of *covariant* derivation as given by

$$\vec{\xi} \cdot \nabla = \vec{\xi} \cdot \partial + \{\vec{\xi} \cdot \Gamma\} \tag{26}$$

where $\{\vec{\xi} \cdot \Gamma\}$ is purely algebraic operator involving contractions with 2-index quantity $(\vec{\xi} \cdot \Gamma)^{\mu}_{\nu} = \xi^{\rho} \Gamma^{\mu}_{\rho\nu}$, as exemplified, for a vectorial (e.g., Killing) field k^{μ}, or a covectorial (e.g., Maxwellian) form $\dot{A}_\mu$, by

$$\{\vec{\xi}\cdot\Gamma\}k^\mu = (\vec{\xi}\cdot\Gamma)^\mu_{\ \nu}k^\nu, \qquad \{\vec{\xi}\cdot\Gamma\}A_\mu = -(\vec{\xi}\cdot\Gamma)^\nu_{\ \mu}A_\nu. \tag{27}$$

Alternatively, instead of using the connection-dependent covariant derivative, it may be more appropriate to work with the corresponding *Lie derivative*, as given by a prescription of the form

$$\vec{\xi}\pounds = \vec{\xi}\cdot\nabla - \{\nabla\xi\}, \tag{28}$$

in which the operator $\{\nabla\xi\}$ acts by contractions with the displacement gradient tensor $\nabla_\nu\xi^\mu$, in the manner exemplified respectively for a vector k^μ, or a 1-form (i.e., covector) A_μ, by the formulae

$$\{\nabla\xi\}k^\mu = k^\nu\nabla_\nu\xi^\mu, \qquad \{\nabla\xi\}A_\mu = -A_\nu\nabla_\mu\xi^\nu. \tag{29}$$

It can be seen that connection cancels out, so that the prescription (28) will be equivalently expressible in terms just of partial derivative components $\partial_\nu\xi^\mu$ as

$$\vec{\xi}\pounds = \vec{\xi}\cdot\partial - \{\partial\xi\}. \tag{30}$$

Another kind of variation that is particularly important in the context of brane mechanics—because (unlike the Eulerian, covariant, and Lie derivatives) it is always well defined even for fields whose support is confined to the brane worldsheet—is what is known as the *Lagrangian variation*, meaning change with respect to background coordinates that are dragged by displacement. In the case of a field that is not confined to the brane worldsheet, so that its Eulerian variation is well defined, this latter kind will be related to the corresponding Lagrangian variation by the well-known Lie derivation formula

$$\underset{\mathrm{L}}{\delta} = \underset{\mathrm{E}}{\delta} + \vec{\xi}\pounds. \tag{31}$$

Yet another possibility that may be useful is to express the *Eulerian* (fixed background point) variation in the form

$$\underset{\mathrm{E}}{\delta} = \underset{\Gamma}{\delta} - \vec{\xi}\cdot\nabla, \tag{32}$$

where *parallely transported variation* is defined—not just for background field, but also for tensor confined to brane—by

$$\underset{\Gamma}{\delta} = \delta + \{\vec{\xi}\cdot\Gamma\}, \tag{33}$$

using operator notation introduced above.

Unlike the covariant and Lie derivations $\vec{\xi}\cdot\nabla$ and $\vec{\xi}\pounds$ and unlike the Eulerian variation $\underset{\mathrm{E}}{\delta}$, the *parallel* variation $\underset{\Gamma}{\delta}$ shares with the *Lagrangian* variation $\underset{\mathrm{L}}{\delta}$ the important property of being well defined not just for background fields but also for fields whose support is confined to the brane worldsheet. The *Lagrangian* variation

$\underset{\mathrm{L}}{\delta}$ will always be expressible directly in terms of the corresponding *parallel* variation $\underset{\Gamma}{\delta}$ by a relation of the form

$$\underset{\mathrm{L}}{\delta} = \underset{\Gamma}{\delta} - \{\nabla\vec{\xi}\}, \tag{34}$$

in which it can be seen that connection dependence cancels out, leaving an expression of the simple form $\{\nabla\vec{\xi}\}$

$$\underset{\mathrm{L}}{\delta} = \delta - \{\partial\vec{\xi}\}, \tag{35}$$

where the action of the algebraic operator $\{\partial\vec{\xi}\}$ is exemplified for a vector k^μ, or a covector A_μ, by the respective formulae

$$\{\partial\vec{\xi}\}k^\mu = k^\nu\partial_\nu\xi^\mu, \qquad \{\partial\vec{\xi}\}A_\mu = -A_\nu\partial_\mu\xi^\nu. \tag{36}$$

In conclusion of this overview of the relationships between the various kinds of infinitesimal variations that are commonly useful, it is to be mentioned that in literature dealing with purely nonrelativistic contexts in which it is possible (though not necessarily wise) to work exclusively with space coordinates of strictly Cartesian (orthonormal) type, the variations of the kind referred to here as "parallel" are generally described as "Lagrangian" by many authors. That usage does not necessarily lead to confusion, because for scalars the distinction does not arise, and because such authors systematically eschew the use (and the technical advantages) of Lagrangian variations of the fully comoving kind (that is considered here) by working exclusively with tensor components that are evaluated in terms only of orthonormal frames.

6. EVALUATION IN TERMS OF LAGRANGIAN VARIATIONS

In typical applications, the relevant set of configuration components q^{A} will include a set of brane field components φ^α as well as the background coordinates x^μ, so that in terms of displacement vector $\xi^\mu = \delta x^\mu$ the Liouville current will take the form

$$\Theta^\nu = \|\eta\|^{-1/2}x^\nu_{,i}\left(p^i_\alpha\delta\varphi^\alpha + p^i_\mu\xi^\mu\right) = \pi^{\ \nu}_\alpha\delta\varphi^\alpha + \pi^{\ \nu}_\mu\xi^\mu, \tag{37}$$

in which the latter version replaces the original momentum components by corresponding background tensorial momentum variables that are defined by

$$\pi^{\ \nu}_\alpha = \|\eta\|^{-1/2}x^\nu_{,i}\,p^i_\alpha. \tag{38}$$

and

$$\pi^{\ \nu}_\mu = \|\eta\|^{-1/2}x^\nu_{,i}\,p^i_\mu. \tag{39}$$

To obtain an analogously tensorial formula for the symplectic current 2-form, it is convenient, as a first step, to take advantage of the symmetry property $\Gamma^{\nu}_{\mu\rho} = \Gamma^{\nu}_{\rho\mu}$, of the Riemannian connection of the background space–time metric, which allows substitution of *parallel* variation $\underset{\Gamma}{\delta}\, p^i_\mu = \delta p^i_\mu - \Gamma^{\nu}_{\mu\rho} p^i_\nu \xi^\rho$ for δp^i_μ so as to provide an expression of the form

$$\Omega^\nu = \|\eta\|^{-1/2} x^\nu_{,i} \big(\delta p^i_\alpha \wedge \delta\varphi^\alpha + \underset{\Gamma}{\delta}\, p^i_\mu \wedge \xi^\mu\big). \tag{40}$$

The next step is to evaluate the relevant momentum variations in terms of the corresponding *Lagrangian* variations, using the formulae

$$\|\eta\|^{-1/2} x^\nu_{,i} \delta p^i_\alpha = \underset{\mathrm{L}}{\delta}\, \pi^\nu_\alpha + \pi^\nu_\alpha \overline{\nabla}_\rho \xi^\rho, \tag{41}$$

and

$$\|\eta\|^{-1/2} x^\nu_{,i}\, \underset{\mathrm{L}}{\delta}\, p^i_\mu = \underset{\mathrm{L}}{\delta}\, \pi^\nu_\mu - \pi^\nu_\rho \nabla_\mu \xi^\rho + \pi^\nu_\mu \overline{\nabla}_\rho \xi^\rho. \tag{42}$$

The advantage of *Lagrangian variations* is their convenience for relating the relevant intrinsic physical quantities via the appropriate equations of state.

7. THE SIMPLY ELASTIC CATEGORY

The illustration that follows will be restricted to the *simply elastic* category (including the case of an ordinary barotropic perfect fluid) in which (with respect to a suitably comoving internal reference system σ^i) there are no independent surface fields at all—meaning that the φ^α and the p^i_α are absent—and in which the only relevant background field is the metric $g_{\mu\nu}$ that is specified as a function of the external coordinates x^μ.

In any such simply elastic case, the generic variation of the Lagrangian is fully determined by the relevant surface stress momentum energy density tensor $\overline{T}^{\mu\nu}$ according to the standard prescription $\delta\mathcal{L} = \frac{1}{2}\|\eta\|^{1/2}\overline{T}^{\mu\nu}\, \underset{\mathrm{L}}{\delta}\, g_{\mu\nu}$, whereby $\overline{T}^{\mu\nu}$ is specified in terms of partial derivation of the action density with respect to the metric. In a fixed background (i.e., in the absence of any Eulerian variation of the metric), the Lagrangian variation of the metric will be given, according to the formula (31), by $\underset{\mathrm{L}}{\delta}\, g_{\mu\nu} = \vec{\xi}\pounds g_{\mu\nu} = 2\nabla_{(\mu\nu)}\xi$. By comparing this to canonical prescription $\delta\mathcal{L} = \mathcal{L}_\mu \xi^\mu + p^i_\mu \xi^\mu_{,i}$ with $\xi^\mu = \delta x^\mu$, it can be seen that the relevant partial derivatives will be given by the (nontensorial) formulae $\mathcal{L}_\mu = \|\eta\|^{1/2}\Gamma^\nu_{\mu\rho}\overline{T}^\rho_\nu$ and $p^i_\mu = \|\eta\|^{1/2}\overline{T}_{\mu\nu}\eta^{ij}x^\nu_{,j}$.

The next step is to translate the result into background tensorial form. It can be seen from the preceding work that in the *simply elastic* case, the canonical momentum tensor π^ν_μ and the Liouville current Θ^ν will be given just in terms of surface stress tensor $\overline{T}^{\mu\nu}$ by the very simple formulae

$$\pi_\mu^{\ \nu} = \overline{T}_\mu^{\ \nu}, \qquad \Theta^\nu = \overline{T}_\mu^{\ \nu} \xi^\mu. \tag{43}$$

To proceed, we must consider the second-order metric variation, whereby (following Friedman and Schutz (1975)) the hyper Cauchy tensor (generalized elasticity tensor) $\overline{C}^{\mu\nu\rho\sigma} = \overline{C}^{\rho\sigma\mu\nu}$ is specified (Battye and Carter, 1995) in terms of Lagrangian variations by a partial derivative relation of the form

$$\underset{\mathrm{L}}{\delta}(\|\eta\|^{1/2}\overline{T}^{\mu\nu}) = \|\eta\|^{1/2}\overline{C}^{\mu\nu\rho\sigma} \underset{\mathrm{L}}{\delta} g_{\rho\sigma}. \tag{44}$$

The symplectic current is thereby obtained in the form

$$\Omega^\nu = \left(2\overline{C}_{\mu\rho}^{\ \ \nu\ \sigma} \overline{\nabla}_\sigma \xi^\rho + \overline{T}^{\nu\rho} \overline{\nabla}_\rho \xi_\mu\right) \wedge \xi^\mu. \tag{45}$$

8. THE SIMPLE THE DIRAC–GOTO–NAMBU CASE

The perfectly elastic category to which the formula (45) is applicable includes examples such as the case (to which much attention has been given in recent work on cosmology) of 3-brane world model with a matter content consisting of a barotropic perfect fluid matter.

The consideration of such cases will however be left for future work, while the present paper will be concluded by the treatment of the relatively trivial special case of a Dirac–Goto–Nambu type brane, i.e., a brane on which there are no internal fields at all, so that the Lagrangian scalar L introduced in (2) will simply be a constant, which will be expressible in the form

$$L = -m^{p+1} \tag{46}$$

for some fixed mass scale m.

In terms of the of tangential and orthogonal projectors η_ν^μ and $\perp_\nu^\mu = g_\nu^\mu - \eta_\nu^\mu$, it can be seen that for the Dirac–Nambu–Goto case characterized by (46) the surface stress energy momentum density tensor will be given by an expression of the familiar simple form

$$\overline{T}^{\mu\nu} = -m^{p+1}\eta^{\mu\nu}, \tag{47}$$

while the emphasized Cauchy tensor will be obtained (Battye and Carter, 1995) in the (less well known) form

$$\overline{C}^{\rho\sigma\mu\nu} = m^{p+1}\left(\eta^{\mu(\rho\sigma)\nu}\eta - \frac{1}{2}\eta^{\mu\nu}\eta^{\rho\sigma}\right). \tag{48}$$

It can thus be seen that the canonical symplectic current (45) will be given explicitly by the formula

$$\Omega^\nu = m^{p+1}\left(\eta^{\nu\sigma}\perp_{\mu\rho} + 2\eta_\mu^{[\nu\rho]}\eta_\rho\right)\xi^\mu \wedge \overline{\nabla}_\sigma \xi^\rho. \tag{49}$$

It can now be checked by direct comparison that this canonical symplectic current does indeed agree with the antisymmetric bilinear current that I originally obtained (Carter, 1993) by a rather different approach. The claim by Cartas-Fuentaevilla that this bilinear current is of canonical (closed since exact) type is thereby confirmed.

The reason why the original argument to this effect (Cartas-Fuentevilla, 2002a) was not entirely convincing was that it depended on an assumption to the effect that $\xi^{\nu} \wedge \nabla_{\nu}\xi^{\mu}$ should vanish. The meaning of this condition is that the pair of displacement vector fields involved should commute, something that could always be imposed for a brane in the restricted sense (but not for not for the dimensionally maximal limit case of a space filling fluid or solid medium) by using the gauge freedom to make arbitrary adjustments of the choice of the displacement field off the worldsheet where it has no physical effect. However, instead of being invoked as a (perfectly legitimate) choice of gauge, the commutator was expressed using the dangerously ambiguous abbreviation scheme in which the wedge symbol $\wedge$ was omitted so that it took the form $\xi^{\nu}\nabla_{\nu}\xi^{\mu}$, whose vanishing was accounted for on the basis of a reinterpretation as if the product were of symmetric type, involving just a single displacement vector field ξ^{ν}, which was thereby required to be geodesic. It happens that this (unnecessary and insufficient) condition of geodicity could also (if genuinely needed) be imposed (on one but not both of the commuting vector fields) as a choice of gauge off the worldsheet, but it was unjustifiably alleged (Cartar-Fuentevilla, 2002a) to be implicit as a necessity for my method of analysis (Carter, 1993).

Despite of the fact that it does not have to apply in general, the consideration that the litigious intermediate requirement (namely the simplification provided by the vector commutation condition, not to mention the quite redundant geodicity condition) that was invoked (Carter-Fuentevilla, 2002a) can actually be imposed as an admissible choice of gauge, means that if used more carefully it could after all provide a logically valid chain of reasoning leading to the final (gauge invariant) conclusion—albeit by a route that is less explicit and direct than that of the present paper (which makes no use of any gauge restrictions at all).

To complete this clarification, I would emphasize that my method does not depend on any (geodesic or other) restriction on the choice of the displacement vector field off the worldsheet. (This means that the method is applicable, not just to branes in the restricted sense, but also to ordinary space filling solids and fluids, for which there cannot be any freedom to adjust the displacement field, because no off-shell region is available.) As discussed in more detail in a more recent review (Cartar, 1996), my system of analysis does indeed involved the use of geodicity; however, it is not invoked as a restriction on the infinitesimal displacement field ξ^{ν} but merely as a means of using an (entirely arbitrary) infinitesimal displacement field to specify a corresponding finite displacement.

ACKNOWLEDGMENTS

I thank D. Steer for technical discussions, and E. Gunzig and E. Verdaguer for organizing the Peyresq meeting that was the occasion for undertaking this work.

REFERENCES

Battye, R. A. and Carter, B. (1995). Gravitational Perturbations of Relativistic Membranes and Strings. *Physics Letters B* **357**, 29–35; *Preprint* hep-ph/9508300.

Battye, R. A. and Carter, B. (2000). Second order Lagrangian and symplectic current for gravitationally perturbed Dirac–Goto–Nambu strings and branes. *Classical and Quantum Gravity* **17**, 3325–3334; *Preprint* hep-th/9811075.

Carter, B. (1993). Perturbation dynamics for membranes and strings governed by the Dirac–Goto–Nambu action in curved space. *Physical Review D* **48**, 4835–4838.

Carter, B. (1996). Brane dynamics for treatment of cosmic strings and vortons. In Proceedings of the 2nd Mexican School of Gravitation and Mathematical Physics, ed. A. Garcia, C. Lammerzahl, A. Macias, T. Matos, and D. Nunez. eds. Science Network Publishing, Tlaxcala, Mexico; *Preprint*. hep-th/9705172.

Carter, B. (1997). Electromagnetic self interaction in strings. *Physics Letters B* **404**, 246–252; *Preprint* hep-th/9704210.

Carter, B. and Battye, R. A. (1998). Non-divergence of gravitational self-interactions for Goto–Nambu strings. *Physics Letters B* **430**, 49–53; *Preprint* hep-th/9803012.

Carter, B., Battye, R. A., and Uzan, J. P. (in press). Gradient formula for linearly self-interacting branes. *Communications in Mathematical Physics*; [hep-th/0204042].

Cartas-Fuentevilla, R. (1998). Symplectic current for the field perturbatios in dilaton axion gravity coupled with Abelian fields. *Physical Review D* **57**, 3443–3448.

Cartas-Fuentevilla, R. (2002a). Identically closed two-form for covariant phase space quantisation of Dirac–Nambu–Goto p-branes in a curved space-time. *Physics Letters B* **536**, 283–288; *Preprint* hep-th/0204133.

Cartas-Fuentevilla R. (2002b). Global symplectic potentials in the Witten covariant phase space for bosonc extendons. *Physics Letters B* **536**, 289–293; *Preprint* hep-th/0204135.

Crncovic, C. and Witten, E. (1987). Covariant description of canonical formalism in geometric theories. In *300 Years of Gravitation*, S. W. Hawking, and W. Israel eds., Cambridge University Press, Cambridge, UK, pp. 676–684.

Friedman, J. and Schutz, B. A. (1975). On the stability of relativistic systems. *Astrophysical Journal* **200**, 204–220.

Nutku, Y. (2000). Covariant symplectic structure of the complex Monge–Ampere equation. *Physics Letters A* **268**, 293; *Preprint* hep-th/0004164.

Rovelli, C. (2002). Covariant Hamiltonian formalism for field theory: symplectic structure and Hamilton–Jacobi structure on the space G. *Preprint* gr-qc/0207043.

Soh, K.-S. (1994). Covariant symplectic structure of two-dimensional dilaton gravity. *Physical Review D* **49**, 1906–1911.

Witten, E. (1986). Interacting field theory of open superstrings. *Nuclear Physics B* **276**, 291.

Zuckerman, G. J. (1987). In *Mathematical Elements of String Theory*, S. T. Yau, ed., World Scientific, Singapore.